Prijsvraag extreme waardentheorie

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We want to test the the null hypothesis:

$H_0 : \mu \leq 1000\ (euro)$ (*"portfolio=OK!"*)

against the alternative:

$H_1 : \mu > 1000\ (euro)$ ("alarm, action required!")

based on a significance level $\alpha = 0.05$.

The losses in the insurance portfolio are assumed to have a Pareto distribution with CDF:

$$F(x) = 1 - \frac{1}{(1+x)^\beta}$$

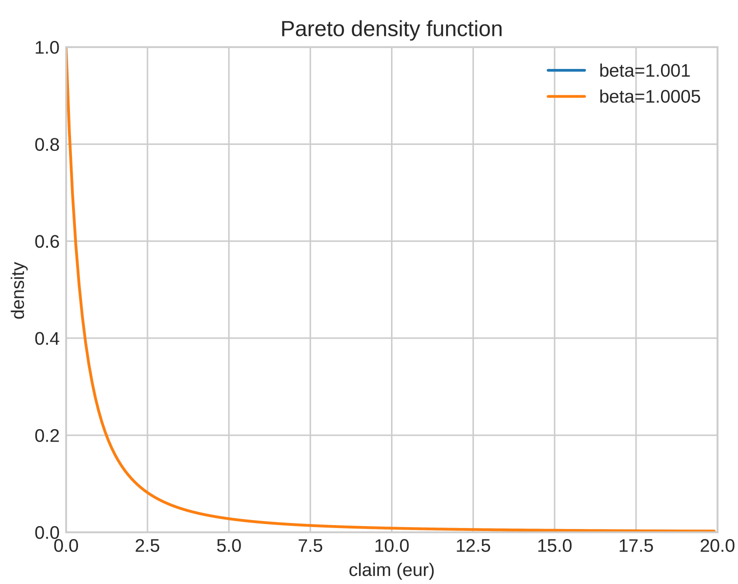
In order to avoid cluttering the main text too much, I will leave some calculation details for the appendix.

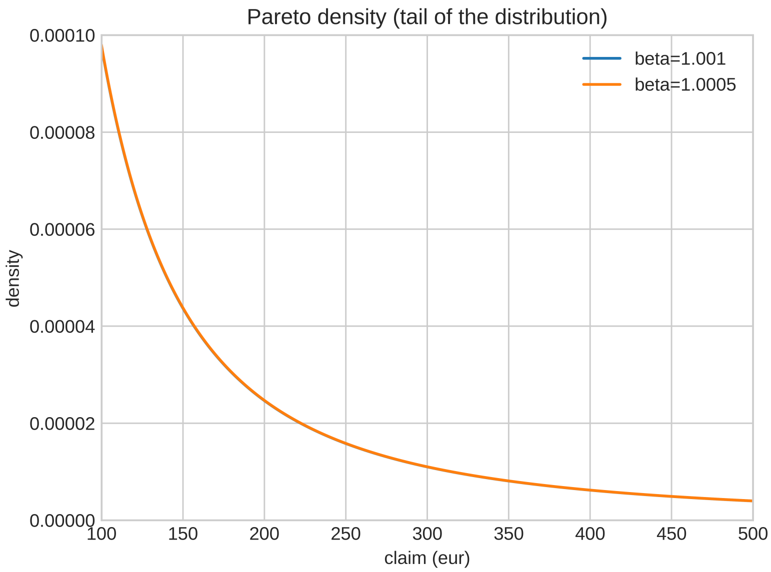
1. Characteristics of the Pareto distribution

The density of the distribution is:

$$f(x) = F'(x) = \frac{\beta}{(1+x)^{\beta+1}}$$

and is highly skewed as shown in the first plot below. It has a very long tail and even in the tail it is hard to see the difference between a Pareto distribution with parameter $\beta = 1.001\ (\mu = 1000)$ and $\beta = 1.0005\ (\mu = 2000)$, as shown in the second plot.





The expected value of the Pareto distribution ($\beta > 1$) is:

$$\mu = E[X] = \frac{1}{\beta - 1}$$

and its variance ($\beta > 2$) is:

$$\sigma^2 = Var[X] = \frac{\beta}{\beta - 2} - \frac{2\beta}{\beta - 1} - \frac{1}{(\beta - 1)^2} + 1$$

We cannot calculate the variance of X for $1 < \beta \leq 1$ algebraically but a simulation confirms what we see in the plots: the variance of X becomes quite large, especially for β values close to 1.

As a result, it will be difficult to distinguish between Pareto($\mu = 1000$) when $\beta = 1.001$ and Pareto($\mu = 2000$) when $\beta = 1.0005$.

2. The threshold value c (question a)

We consider the n=1 test:

Reject H_0 if $X \geq c$

As we are looking for a level $\alpha = 0.05$ test, the threshold value c can be derived as follows:

$$P[Reject\ H_0 \mid H_0\ is\ True] = \alpha$$

$$\iff P[X \geq c \mid \mu = 1000] = \alpha$$

$$\iff P[X \geq c \mid \beta = 1.001] = \alpha$$

$$\iff 1 - F_x(c \mid \beta = 1.001) = \alpha$$

$$\iff \frac{1}{(1 + c)^{\beta}} = \alpha$$

$$\iff \frac{1}{(1 + c)^{1.001}} = 0.05$$

$$\iff c^* = 18.94$$

3. Power of the test (question b)

We assume that the true $\mu = 2000$ (i.e. true $\beta = 1.0005$).

Use the threshold value c^* derived in the section above, to calculate the power of our n=1 test:

$$P[Reject\ H_0 \mid \mu = 2000] = P[X \geq c^* \mid \mu = 2000]$$

$$= P[X \geq c^* \mid \beta = 1.0005] = 1 - F_x(c^* \mid \beta = 1.0005)$$

$$= \frac{1}{(1+c^*)^\beta} = \frac{1}{(1+18.94)^{1.0005}} = 0.05007$$

So we conclude that this test is not able to detect changes of the average loss in our insurance portfolio. This is due to the (extremely) skewed nature of the given Pareto distribution. The "normal" recipe that works for normal-like distributions, namely improving the power by increasing the sample size, is not expected to work here as long as we have to deal with those values of β close to 1 that make the sample statistics used for testing the hypothesis very unstable.

Appendix

In this appendix I bring some stuff together I used to derive the results mentioned in the main text.

Pareto distribution: expected value

$$f(x) = F'(x) = \frac{\beta}{(1+x)^{\beta+1}}$$

and

$$A(x) = xf(x)$$

$$= \frac{\beta x}{(1+x)^{\beta+1}} = \frac{\beta}{(1+x)^\beta} - \frac{\beta}{(1+x)^{(\beta+1)}}$$

$$= \beta\left((1+x)^{-\beta} - (1+x)^{-\beta-1}\right)$$

therefore for $\beta > 1$ we have:

$$E[X] = \int_0^\infty A(x)dx$$

$$= \beta(\frac{(1+x)^{-\beta+1}}{-\beta+1} + \frac{(1+x)^{-\beta}}{\beta})\Big|_0^\infty$$

$$= -\beta\,(\frac{1}{-\beta+1} + \frac{1}{\beta})$$

$$= \frac{\beta}{\beta-1} - 1 = \frac{1}{\beta-1}$$

Pareto distribution: variance

$$Var[X] = E[X^2] - (E[X])^2$$

and

$$B(x) = x^2 f(x)$$

$$= \frac{\beta x^2}{(1+x)^{\beta+1}} = \frac{\beta}{(1+x)^{\beta+1}} - \frac{2\beta}{(1+x)^\beta} + \frac{\beta}{(1+x)^{\beta-1}}$$

therefore:

$$E[X^2] = \int_0^\infty B(x)dx$$

$$= \beta(\frac{(1+x)^{-\beta}}{-\beta} - 2\frac{(1+x)^{-\beta+1}}{-\beta+1} + \frac{(1+x)^{-\beta+2}}{-\beta+2})\Big|_0^\infty$$

$$= 1 + \frac{2\beta}{1-\beta} - \frac{\beta}{2-\beta}$$

so for $\beta > 2$ we have:

$$Var[X] = E[X^2] - (E[X])^2$$

$$= 1 + \frac{2\beta}{1-\beta} - \frac{\beta}{2-\beta} - \frac{1}{(\beta-1)^2}$$

$$= \frac{\beta}{\beta-2} - \frac{2\beta}{\beta-1} - \frac{1}{(\beta-1)^2} + 1$$