BY ALEXANDER VAN HAASTRECHT AND ANTOON PELSSER

RISK MANAGEMENT

Managing Simulation Uncertainty in Percentiles and Value-at-Risk

Accurate estimates of simulation uncertainty around percentiles and Value-at-Risk (VaR) measures are important for various practical applications in insurance, pensions and banking.

Managing simulation uncertainty requires using an adequate number of scenarios to produce reliable risk estimates. The simplest approach for determining simulation uncertainty would be to reperform the calculations many times (with a different seed or via bootstrapping) and measure the variation in the outcomes of the re-simulated estimates. However, given that in practice often 100,000 scenarios or more are used, this approach is quite impractical.

In this article we describe an alternative method based on an 'in-sample' estimate of the simulation error. This 'in-sample' estimate relies solely on the original simulations to determine the uncertainty around the percentile, making it practical to compute.

Dr. A. van Haastrecht (left) is a partner at Risk at Work and holds a position as lecturer at the Finance & Risk Management department at the VU Amsterdam.

Prof. dr. A.A.J. Pelsser Hon FIA is a partner at Risk at Work and holds a position as professor in Finance and Actuarial Science at the University of Maastricht.

UNDERSTANDING PERCENTILES AND CONFIDENCE INTERVALS

A percentile is a measure that indicates the value below which a given percentage of observations in a group falls. For example, the 90th percentile of a dataset is the value below which 90% of the observations lie. Often value-at-risk or expected shortfall risk measures are applied in finance to calculate capital requirements, and for pensions to provide insights in future outcomes.

A confidence interval provides a range of values that likely contain the population parameter (in this case, the percentile) with a specified level of confidence. For instance, a 95% confidence interval for the 90th percentile means we are 95% confident that the true 90th percentile falls within this range.

THEORICAL RESULT

We have a theoretical result regarding the probability distribution of order statistics, as detailed in [1] and [2]. For a uniform distribution, as the sample size N tends to infinity, the p-quantile (or the value below which p% of the data falls) becomes asymptotically normally distributed with mean p and variance $\frac{p(1-p)}{N}$.

Although our data isn't uniformly distributed, sorting the sample allows us to treat the ranks as if they were from a uniform distribution scaled by the sample size N. Hence, the rank numbers of the sorted sample represent a scaled uniform order statistic which convergences asymptotically to a normal distribution with mean $N \cdot p$ and variance $N \cdot p \cdot (1-p)$.

Therefore, by sorting our original data and considering the ranks, we can leverage on this theoretical result for estimating percentiles and their variability. For instance, to estimate the p-th percentile in our data, we can use the value corresponding to the rank $N \cdot p$ in the sorted sample. Additionally, we can use the standard error $\sqrt{N \cdot p \cdot (1-p)}$ of the rank numbers to construct a confidence interval for quantifying the uncertainty around our percentile estimate.

STEPS TO CONSTRUCT THE CONFIDENCE INTERVAL

The steps to construct the confidence interval are as follows:

1. **Sample size and percentile**: Let N be the number of simulations and p be the desired percentile (e.g. $p = 0.005$ or the 0.5^{th} percentile)
2. **Sort the data**: let X_1 denote the smallest outcome, X_2 the second smallest outcomes, and so on until the largest outcome X_N.
3. **Percentile Calculation**[1]: Calculate the percentile by taking the value X_k, with rank $k = N \cdot p$
4. **Critical value**: For a desired confidence level α (e.g. $\alpha = 0.05$ for 95% confidence interval), the critical value z is derived from the standard normal distribution as $z = N^{-1}(1 - \alpha/2)$. For 95% confidence interval $z \approx 1.96$.
5. **Confidence interval**: Calculate the confidence interval width for the rank numbers as $\Delta = ceil(z_\alpha \cdot \sqrt{N \cdot p \cdot (1-p)})$ and construct the confidence interval bounds $X_{k-\Delta}$ and $X_{k+\Delta}$ around the p^{th} percentile X_k.

Example Calculation

Insurers are often interested in the uncertainty around the 0.5% percentile. Suppose that 100,000 scenarios are used, one can then estimate the 0.5%-percentile by taking the value with rank $100{,}000 \cdot 0.5\% = 500$ from the sorted sample.

Using the asymptotic distribution result, one can construct a 95% confidence interval width for the rank numbers as $\Delta = 1.96 \cdot \sqrt{100{,}000 \cdot (1 - 0.5\%) \cdot 0.5\%} = 43.7$, which we round up to 44.

Hence, the simulation uncertainty for the 0.5% percentile in a Monte Carlo sample of size $N = 100{,}000$ can be estimated by taking the values with ranks $500 - 44 = 456$ and $500 + 44 = 544$ as bounds for the 95% confidence interval (illustrated in the figure below).

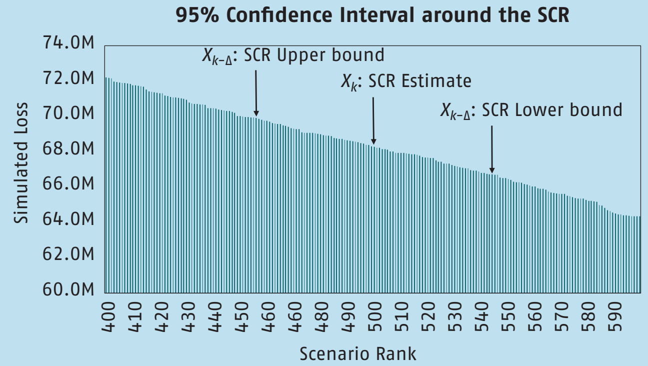

Figure 1: 95% Confidence interval for the SCR can be constructed from the scenarios with ranks $k - \Delta$ (scenario 456) and $k + \Delta$ (scenario 544).

CASE STUDY: CREDIT RISK CAPITAL

In order to illustrate the method, we apply it to credit risk capital calculations. Management here aims to ensure the number of simulations is sufficient to keep uncertainty around the capital estimate below a predefined threshold.

To this end, we assess the credit risk capital's simulation uncertainty across different numbers of scenarios, with the results shown in the figure below.

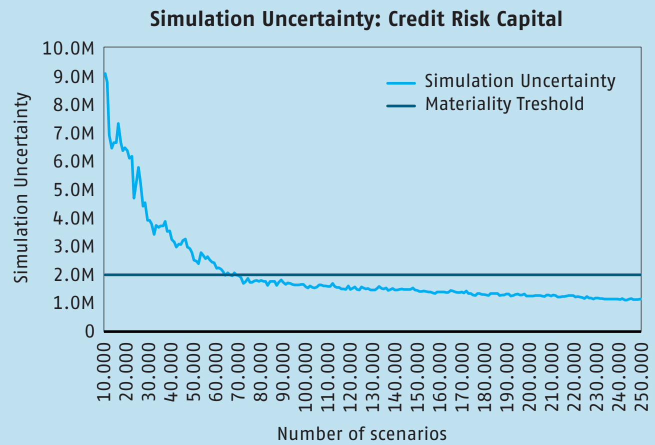

Figure 2: Simulation uncertainty for credit risk capital, as function of the number of scenarios

Based on these outcomes, a minimum of 69,000 scenarios is necessary to achieve a simulation uncertainty below the set materiality threshold. To ensure robustness, management opts for 100,000 scenarios, enhancing risk management and decision-making.

IMPACT OF NUMBER OF SCENARIOS ON SIMULATION UNCERTAINTY

When analyzing the relationship between simulation uncertainty and the number of scenarios using log-scales, an important pattern emerges: the simulation uncertainty decreases proportionally to the square root of the number of scenarios N, see the figure below.

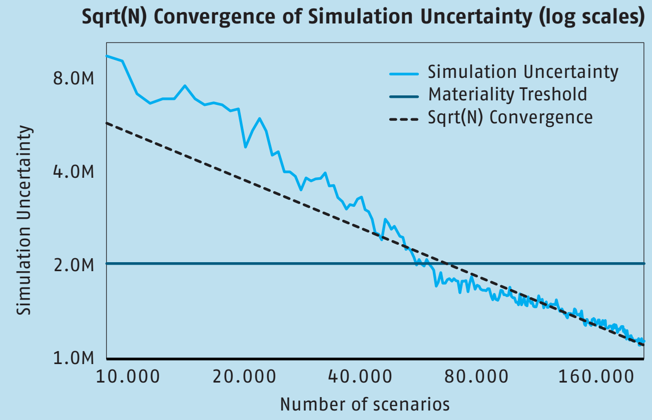

Figure 3: Log-log scale for the simulation uncertainty as function of the number of scenarios.

This relationship implies that to achieve an estimate with half the uncertainty, you must quadruple the number of scenarios. Conversely, if a lower number of scenarios would be used (e.g. due to limited computational resources), the error in the estimates increases accordingly.

This square root dependency highlights the diminishing returns of increasing the number of scenarios: while more scenarios reduce uncertainty, achieving a significantly higher accuracy requires exponentially more scenarios.

CONCLUSION

This article presents an efficient approach to manage simulation uncertainty for percentiles and VaR. By focusing on in-sample estimates and leveraging theoretical properties of order statistics, practitioners can measure the uncertainty of simulated risk estimates without the need for extensive re-sampling.

This method can be beneficial for practitioners in various fields:

1. **Insurance**: Partial internal models used by insurance companies benefit from accurate percentile estimates, enhancing the precision of market, life, and non-life SCR calculations.
2. **Pensions and Asset Liability Management**: In the report of the *Advies Commissie Parameters* (2022, p71-72) this method is used to estimate the 5% and 95% percentiles of accumulated pension at retirement, improving risk assessment and decision-making under various economic scenarios.
3. **Banking**: For various capital calculations, accurate simulation uncertainty estimates lead to more reliable risk assessments and capital requirements. ■

References

[1] F. Mosteller, On Some Useful 'Inefficient' Statistics, Annals of Mathematical Statistics 17(4), December 1946.
[2] Wikipedia order statistic: https://en.wikipedia.org/wiki/Order_statistic
[3] F. de Vries et al. Advies Commissie Parameters, 29 November 2022.

1 – For more advanced calculations, percentiles can also be determined using interpolation between adjacent values, which allows for a (slightly) more precise estimate. The described method can be generalized to incorporate these interpolation techniques.