Implementing external variables in Scenario Sets

Economic scenario sets can be incredibly valuable in tackling uncertainties in the financial world. Scenario sets can give great insights in the possible future outcomes and allow practitioners to determine a strategic approach to various uncertainties. Even though the added value of a scenario set is clear, it can be incredibly difficult to create a scenario set of high quality. In literature, there is a significant amount of research related to generating economic scenario sets. There exist various well described one dimensional models such as the Black-Scholes and the Vasicek model, but both of these models have their own drawbacks. One of the pitfalls of both is that they assume constant volatility and assume no interaction with other macro-economic factors such as inflation. The last two years have painfully demonstrated that volatility in financial markets is variable over time and that macro-economic factors such as inflation impact a broad range of economic variables. In this article, I show how to implement stochastic volatility and external factors such as inflation to well described models. The result of this extension is that the accuracy of the scenario set increases, yielding better insight in potential future outcomes.

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I consider economic scenario sets consisting of forecasted valuations of various broadly-defined asset classes such as fixed income, equity and real estate. In order to obtain forecasted valuations for these asset classes, a state variable for each of the asset classes needs to be modelled over time. Even though these models might differ for each asset class, for the sake of convenience, I consider a general model where the model parameters can be adapted for each asset class specifically. The purpose of this article is to show how I can add variables to a general one dimensional model as I like. In the general model, I assume a state vector X_t which consists of the state variable for the asset class and additional variables of choice e.g. volatility and inflation. For the basic form of the model, I assume a stochastic differential equation (SDE) with drift $\mu(X_t)$, volatility $\sigma(X_t)$, and Brownian motion W_t. Consequently, I impose a general stochastic differential equation of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \,.$$

In case of a multidimensional state vector X_t, the Brownian motion W_t in the equation above will be multivariate. As is done in many financial models, for the sake of convenience and tractability, I impose a general affine structure on the drift $\mu(X_t)$ and volatility $\sigma(X_t)$ given by

$$\mu(X_t) = K_0 + K_1 X_t \,,$$
$$\sigma(X_t)\sigma(X_t)' = H_0 + H_1 X_t \,.$$

Note that X_t, K_0, K_1, H_0, and H_1 are all vectors with a dimension equal to the number of state variables. Also note that for certain values of K_0, K_1, H_0, and H_1, I can obtain the well described Black-Scholes and Vasicek model.

The quality of a scenario set is highly dependent on the accuracy of the model parameters. A model which estimates model parameters inaccurately will be inherently flawed. In case of a one dimensional model, the Maximum Likelihood Estimation (MLE) method can be used to derive an estimator with desirable statistical properties such as consistency and efficiency. In the case of certain multidimensional models however, the closed-form expression of the density functions is not always known. This makes using the MLE significantly more difficult and as a result alternative methods are preferred.

One such alternative method is the Generalised Method of Moments (GMM). GMM can be used to derive an estimator which is consistent and is the most efficient estimator in the class of estimators based on moment conditions. In order to derive the GMM estimator, a practitioner derives the characteristic function, from which moment conditions can be easily obtained. While the density function may not be known in closed-form, the characteristic function can be found in a straight forward way. In the remainder of this article, I show how to derive the expression for the characteristic function and use the characteristic function to derive the moment conditions for GMM.

Under the general model in this article and the imposed affine functions, Duffie, Pan, and Singleton (2000) have shown that the form of the characteristic function is given by

$$\psi(u; (X_t) = e^{C(u,t) + D(u,t)X_t} \,,$$

where the values for $C(u,t)$ and $D(u,t)$ are found by solving the complex valued Ordinary Differential Equations (ODE) below

$$\frac{\partial D(u,t)}{\partial t} = -K_1' D(u,t) - \frac{1}{2} D(u,t)' H_1 D(u,t) \,,$$

$$\frac{\partial D(u,t)}{\partial t} = -K_0' D(u,t) - \frac{1}{2} D(u,t)' H_0 D(u,t) \,,$$

with boundary solutions $D(u,t)$ = u and $C(u,t) = 0$. Note that in the expression, $C(u,t)$ is a scalar and in case of multivariate models $D(u,t)$ is a vector with dimensions equal to the number of variables. By solving this ODE, I obtain the closed-form expression for the characteristic function of a multivariate SDE in a straightforward way. Furthermore, it can be shown that from the characteristic function, the k-th moment can be derived using the relation below

$$E[X^k] = \frac{1}{i^k} \left. \frac{\partial^k \phi(u; x_t)}{\partial u^k} \right|_{u=0} .$$

Using the relation above, I directly obtain the k-th moment condition of any stochastic differential equation. By deriving at least as many moment conditions as there are model parameters, I use the empirical GMM method to estimate the model parameters in the general model. The choice of empirical data is specific to each asset class resulting in different parameter values. Note that some external variables, such as the volatility, might be latent and would require a proxy to obtain historical values. Once the model parameters are estimated using GMM, I use the forward Euler Method to generate forecasted valuations of asset classes such as fixed income, equity and real estate. The horizon of the scenario set and the number of scenarios can be freely chosen as part of this method allowing for a flexible and convenient use.

The method described in this article can be used to derive moment conditions of any multivariate stochastic model. Due to the complexity of multivariate models, the closed-form expression of the density functions is not always known, making the MLE method difficult. As an alternative method, I propose a method based on GMM and the characteristic function. Even though the density function may not be known in closed-form, the characteristic function can be found in a straight forward way. After obtaining the closed-form expressions of the moment conditions using the characteristic function, I estimate the model parameters using the empirical GMM method based on financial data for each asset class. Once the parameters are estimated, I can use the model to simulate valuations of the asset classes as I like. ■

References

Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusion. *Econometrica* 68(6), 1343–1376.